MATH2050C Assignment 11

Deadline: April 8 , 2025.

Hand in: 5.4 no. 3, 4, 7; Suupl. Problems no 1, 2.

Section 5.4 no. 3, 4, 6, 7, 8, 10, 15.

Supplementary Problems

1. Let function f on A satisfy the condition: There is some constant C and $\alpha > 0$ such that $|f(x) - f(c)| \leq C|x - c|^\alpha$ for all $x, c \in A$. Show that f is uniformly continuous on A. (It is called the Lipschitz condition when $\alpha = 1$.)

2. Let $f \in C[1, \infty)$ satisfy $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show that f is uniformly continuous on $[1, \infty)$.

3. Let f be a uniformly continuous function on $[0, \infty)$. Show that there is a constant C such that $|f(x)| \leq C(1 + x)$.

See next page

Uniform Continuity of Functions

Let f be continuous on some nonempty set A in $\mathbb{R}$. When f is continuous at some $c \in A$, it means for each $\varepsilon > 0$, there is some δ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in A, |x - c| < \varepsilon$. Here δ in general depending on c and ε. Now, f is said to be *uniformly continuous* on $E \subset A$ if for each $\varepsilon > 0$, there is a δ such that $|f(x) - f(c)| < \varepsilon$ for all $x \in E, |x - c| < \delta$.

Example 11.1 The function $f(x) = 1/x$ is continuous but not uniformly continuous on $(0, \infty)$. Let $c \in (0, \infty)$ The preimage of f on $(f(c) - \varepsilon, f(c) + \varepsilon)$ is $(c/(1 + \varepsilon c), c/(1 - \varepsilon c))$. Since $c/(1+\varepsilon c) < c/(1-\varepsilon c)$, the optimal δ so that $|f(x)-f(c)| < \varepsilon$, for $x, |x-c| < \delta$ is $\delta^* = c/(1+\varepsilon c)$. It clearly depends on c and ε. As $\delta^* \to 0$ as $c \to 0^+$, we cannot choose a uniform δ so that $|f(x) - f(c)| < \delta$ for all $x, c, |x - c| < \delta$, hence f is not uniformly on $(0, \infty)$.

Example 11.2 The function $g(x) = x^2$ is not uniformly continuous on $[1, \infty)$. For, the preimage of g on $(g(c) - \varepsilon, g(c) + \varepsilon)$ is $(\sqrt{c^2 - \varepsilon}, \sqrt{c^2 + \varepsilon})$. One checks that $c - \sqrt{c^2 - \varepsilon} > \sqrt{c^2 + \varepsilon} - c$ hence the optimal δ is $\delta^* = \sqrt{c^2 + \varepsilon} - c = \varepsilon/(\sqrt{c^2 + \varepsilon} + c)$ which tends to 0 as $c \to \infty$. Therefore, g is not uniformly continuous.

Example 11.3 The function $h(x) = \sin 1/x$ is not uniformly continuous on $(0, 1]$. For, in case it is uniformly continuous, for $\varepsilon = 1/2$, we can find a δ such that $|f(x) - f(y)| < 1/2$ whenever x, y belongs to an open interval of length less than δ. For large n, the points $x_n = 1/2n\pi, y_n = 1/(2n\pi + \pi/2)$ both belong to $(0, \delta)$ but $|f(x_n) - f(y_n)| = 1 > 1/2$, contradiction holds. Hence h is not uniformly continuous on $(0, 1]$.

Theorem 11.1 Every continuous function on $[a, b]$ is uniformly continuous.

We refer to the textbook for a proof. Note that the same proof works for all dimensions where the theorem states as, every continuous function on a closed, bounded set in $\mathbb{R}^n$ is uniformly continuous.

Theorem 11.2 Every uniformly continuous function on (a, b) has a unique uniformly continuous extension to $[a, b]$.

Again we refer to the textbook for a proof. The key observation is that a uniformly continuous function maps a Cauchy sequence to a Cauchy sequence. The higher dimensional version of this theorem is that every uniformly continuous function on $A \subset \mathbb{R}^n$ has a unique uniformly continuous extension to $\overline{A}$ where $\overline{A}$ is the union of A and its cluster points.

In the following paragraphs we would like to use the notion of the oscillation of a function to study uniform continuity. Although most results are valid in all dimensions, we focus on dimension one.

Let E be a nonempty set in $\mathbb{R}$ and f a bounded function on E. The *oscillation of f over E* is defined to be

$$\text{osc}_E f = \sup_E f - \inf_E f = \sup_{x,y \in E} |f(x) - f(y)|.$$

Theorem 11.3 (Oscillation Theorem) A bounded function f is uniformly continuous on a set E if and only if, given $\varepsilon > 0$, there is some δ such that on every (open or closed) interval I of length δ, $\text{osc}_{I \cap E} f \leq \varepsilon$.

Proof When f is uniformly continuous, for each $\varepsilon > 0$, there is some δ such that $|f(x) - f(y)| < \varepsilon, x, y \in E, |x-y| < \delta$. Hence when $x, y \in I \cap E$ where the open interval I has length δ, $|x-y| < \delta$ and $|f(x) - f(y)| < \varepsilon$. Hence, taking sup over all $x, y \in I \cap E$, we conclude $\text{osc}_{I \cap E} f \leq \varepsilon$. Conversely, taking $\varepsilon/2 > 0$, there is some δ such that $\text{osc}_{I \cap E} f \leq \varepsilon/2$ whenever I if of length δ. When x, y satisfy $|x - y| < \delta$, we can find such an interval I containing x, y. Therefore, $|f(x) - f(y)| \leq \text{osc}_{I \cap E} f \leq \varepsilon/2 < \varepsilon$.

Example 11.2' The function $f(x) = x^2$ is not uniformly on $[0, \infty)$. Let us look at a subinterval of the form $I = (x_0, x_0 + \delta)$. Since this function is increasing $\text{osc}_I f = (x_0 + \delta)^2 - x_0^2 = 2\delta x_0 + 4\delta^2$ which tends to infinity as $x_0 \to \infty$. By Theorem 3, it cannot be uniformly continuous on $[0, \infty)$.

Example 11.3' The function $\sin 1/x$ is not uniformly continuous on $(0, 1]$. Why? Let look at the subinterval $I = (0, \delta)$. No matter how small $\delta > 0$ is, $\text{osc}_I f = 2$. By Theorem 11.3 (taking $\varepsilon < 2$) it cannot be uniformly continuous.

The oscillation of a function on a set can be localised to give the oscillation of a function at a point. Indeed, for f on E and $c \in E$, define the oscillation of f at c to be

$$\omega_f(c) = \lim_{\delta \to 0^+} \text{osc}_{I_\delta \cap E} f = \inf_{\delta > 0} \text{osc}_{I_\delta \cap E} f \ , \quad I_\delta = (c - \delta, c + \delta).$$

Note that the oscillation decreases as δ shrinks, hence the limit always exists and is equal to the infimum. It is easy to see that f is continuous at c iff $\omega_f(c) = 0$.

Monotone Functions

A function is increasing (resp. decreasing) on an interval I if $f(x) \leq f(y)$ (resp. $f(x) \geq f(y)$) whenever $x < y$ in I. It is strictly increasing(resp. strictly decreasing) if $f(x) < f(y)$ (resp. $f(x) > f(y)$) whenever $x < y$ in I. It is clear that f is increasing (resp. strictly increasing) if and only if $-f$ is decreasing (resp. strictly decreasing).

Theorem 11.4 Let f be monotone on the interval I and c an interior point of I. Then the right and left limits always exist at c.

See textbook for a proof. Consequently a monotone function is continuous at c if and only if $\lim_{x \to c^-} f = \lim_{x \to c^+} f$. (Since f is monotone, $f(c)$ is pinched between the two one-sided limits. Hence $f(c) = \lim_{x \to c^-} f$.) If f is defined at the left endpoint a, then $\lim_{x \to a^+} f$ exists and f is continuous at a if and only if $\lim_{x \to a^+} f = f(a)$. A similar situation holds at the right endpoint.

Theorem 11.5 The discontinuity set of a monotone function is countable.

Proof Let's us assume f is increasing on $[a, b]$. For $c \in (a, b)$, clearly $\omega_f(c) = \lim_{x \to c^+} f - \lim_{x \to c^-} f$, so $\omega_f(c) > 0$ iff c is a point of discontinuity of f. Let D be the set of discontinuity of f in (a, b). We have the decomposition $D = \bigcup_k D_k$ where $D_k = \{x \in (a, b) : \omega_f(x) \geq 1/k\}$. We claim: Each D_k contains not more than $k(f(b) - f(a))$ many points. Since the countable union of a finite set is countable, D is countable.

Let $c_1 > c_2 > \cdots > c_N$ be points in (a, b). In the following we take $N = 2$ for simplicity. We have

$$\begin{aligned} f(b) - f(a) &= f(b) - \lim_{x \to c_1^+} f + \lim_{x \to c_1^+} f - \lim_{x \to c_1^-} f + \lim_{x \to c_1^-} f - f(a) \\ &= f(b) - \lim_{x \to c_1^+} f + \omega_f(c_1) + \lim_{x \to c_1^-} f - f(a) \\ &= (f(b) - \lim_{x \to c_1^+} f) + \omega_f(c_1) + (\lim_{x \to c_1^-} f - \lim_{x \to c_2^+} f) + \omega_f(c_2) + (\lim_{x \to c_2^-} f - f(a)) \\ &\geq \omega_f(c_1) + \omega_f(c_2) \ , \end{aligned}$$

since the three terms in brackets are non-negative. In general, we have

$$f(b) - f(a) \geq \sum_{i=1}^{N} \omega_f(c_i) \ .$$

Now, if we have N many points in D_k, $f(b) - f(a) \geq \sum_{i=1}^N \omega_f(c_i) \geq \sum_{i=1}^N 1/k = N/k$, hence $N \leq k(f(b) - f(a))$.

The discontinuity set of f on $[a, b]$ is D and possibly including the endpoints, so it is countable. Now, consider f is defined on (a, b). Observing $(a, b) = \bigcup_j [a+1/j, b-1/j]$, its discontinuity set in (a, b) is also countable since the discontinuity set restricted to each $[a+1/j, b-1/j]$ is countable.